Homework 2—Optimization Models in Finance

2.1 (10 points) A rectangle-shaped cardboard box (with six faces) for packing quantities of small foam balls is to be manufactured. The top, bottom, and front faces must be of double weight (two pieces of cardboard). A problem posed is to find the dimensions of such a box that maximize the volume for a given amount of cardboard, equal to 72 square feet.

2.2 (10 points) Use the optimality conditions to find all local solutions to the problem

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) = x_1 + 2x_2 \\ \text{subject to} & (x_1 - 1)^2 + x_2^2 \le 2 \\ & (x_1 + 1)^2 + x_2^2 \ge 2. \end{array}$$

2.3 (10 points) Let

$$f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3.$$

Find the Newton direction at the point $(x_1, x_2, x_3) = (1, 1, 1)$. What if this is to be done under the constraint $2x_1 + x_2 = 3$?

2.4 (10 points) Two players in the game of Two-Finger Morra simultaneously put out either one or two fingers. Each player must also announce the number of fingers that he believes his opponent has put out. If neither or both players correctly guess the number of fingers put out by the opponent, the game is a draw. Otherwise, the player who guess correctly wins from the other player the sum (in dollars) of the fingers put out by the two players.

1) Construct the reward table for this game.

2) Formulate the linear program that solves the optimal mixed strategy for each player.

2.5 (10 points) (20 points) Consider the problem

$$\begin{array}{ll} \text{minimize} & \mathrm{x}^2 \\ \text{subject to} & x \ge 1. \end{array}$$

What is the solution to this problem? Formulate and solve the dual problem. Formulate the dual to the dual and show that it is equivalent to the primal.

2.6 (10 points) Find the set of all optimal solutions and all geometric multipliers, and sketch the dual function for the following two-dimensional convex programming problem:

$$\begin{aligned} \min \quad & x_1 \\ \text{s.t.} \quad & |\, x_1 \,| + |\, x_2 \,| \leq 1, \quad (x_1, x_2) \in X = R^2. \end{aligned}$$

2.7 (10 points) Consider the problem

$$\begin{aligned} \min \quad & f(x_1, x_2) = 9x_1 + 3x_2 \\ \text{s.t.} \quad & 5x_1 + x_2 \geq 4, \quad x_1, x_2 = 0, 1. \end{aligned}$$

1) Sketch the set of constraint-cost pairs.
2) Sketch the dual function.
3) Solve the problem and its dual, and relate the solutions to your sketch in 1).

2.8 (10 points) A newspaper stand purchases newspapers for \$0.35 and sells them for \$0.5. Each unsold newspaper has salvage value of \$0.1. Assume the demand follows the uniform distribution between 100 and 300. Find the optimal number of papers to order.

2.9 (10 points) A light manufacturing firm is planning a new factory in a rural part of the western United States. A total of 100 employees are to be hired from the 5 surrounding communities. Table shows the number of (equally) qualified workers available in each community and community location coordinates (in miles).

Planners want to choose a factory site that minimizes total employee travel distance. Assume that travel distance is proportional to straight-line distance from community to factory site. Formulate this location-allocation problem as a constrained nonlinear programming problem.

2.10 (10 points) Jim Matthews, Vice President for Marketing of the J.R. Nickel Company, is planning advertising campaigns for two unrelated products. These two

Table 1: Table for Problem 2.9

	Community				
	1	2	3	4	5
Available	70	15	20	40	30
E-W coordinate	0	10	6	7	2
N-S coordinate	0	3	8	5	4

campaigns need to use some of the same resources. Therefore, Jim knows that his decisions on the levels of the two campaigns need to be made jointly after considering these resource constraints. In particular, letting x_1 and x_2 denote the levels of campaigns 1 and 2, respectively, these constraints are $4x_1 + x_2 \leq 20$ and $x_1 + 4x_2 \leq 20$.

In facing these decisions, Jim is well aware that there is a point of diminishing returns when raising the level of an advertising campaign too far. At that point, the cost of additional advertising becomes larger than the increase in net revenue (excluding advertising costs) generated by the advertising. After careful analysis, he and his staff estimate that the net profit from the first product (including advertising costs) when conducting the first campaign at level x_1 would be $3x_1 - (x_1 - 1)^2$ in millions of dollars. The corresponding estimate for the second product is $3x_2 - (x_2 - 2)^2$. This analysis led to the following quadratic programming model for determining the levels of the two advertising campaigns:

$$
\begin{aligned}
\max \quad & f(x_1, x_2) = 3x_1 - (x_1 - 1)^2 + 3x_2 - (x_2 - 2)^2, \\
\text{s.t.} \quad & 4x_1 + x_2 \leq 20 \\
& x_1 + 4x_2 \leq 20 \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}
$$

(a) Obtain the KKT conditions for this problem.

(b) You are given the information that the optimal solution does not lie on the boundary of the feasible region. Use this information to derive the optimal solution from the KKT conditions.