IMPERIAL COLLEGE LONDON

BEng, MEng and MSc EXAMINATIONS 2019

Part III, Part IV and Advanced Mechanical Engineering

for Internal Students of the Imperial College of Science, Technology and Medicine
This paper is also taken for the relevant examination for the Associateship or Diploma

MACHINE SYSTEM DYNAMICS

Wednesday, 15th May: 14.00 to 17.00

This paper contains SIX questions.
Attempt ALL questions.
Question 1 carries 20 marks and all other questions carry 16 marks. The numbers shown by each question are for your guidance; they indicate approximately how the examiners intend to distribute the marks for this paper.
A Data and Formulæ Book is provided.

This is a CLOSED BOOK Examination

Turn over

1. Figure Q1a shows the schematic of a quarter car model. The vertical vibration characteristics of the car and the unsprung mass are to be modelled using a simple 2 DOF mass-spring-damper model.

(a) Draw free body diagrams of each mass clearly showing all the forces acting on each mass. [2%]

(b) Write down the equations of motion (EOM) of the two masses. [2%]

(c) From the EOMs derive the stiffness and mass matrices of the system. [2%]

(d) Recall two simple checks that you can perform on the derived matrices to verify that they are of the expected form. [2%]

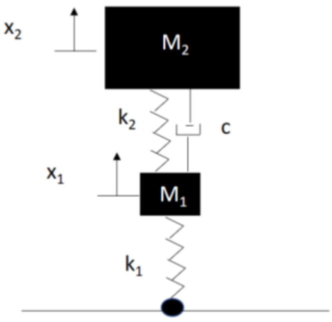
(e) For $k_1 = k_2 = 100$ kN/m, $c = 0$ and M_1=10 kg and M_2= 250 kg calculate the two natural frequencies of the system. [3%]

(f) Calculate and sketch the mode shapes that are associated with the frequencies in (e). [3%]

(g) Sketch the FRF x_1/F_1 for external excitation of the unsprung mass M_1. Write down an equation for the frequency at which this FRF will be minimised and estimate the frequency for a system with parameters as given in (e). [2%]

(h) Motion of the car can be modelled as vertical displacement y of the contact point of spring k_1 with the ground (see figure Q1b). Derive the new EOM that also shows the forcing term due to ground excitation. [2%]

(i) If the car travels over a surface that has a sinusoid shape ($y = \sin x/L$) at what horizontal velocity v would you expect the car body (M_2) to experience maximum displacement if L=1m? [2%]



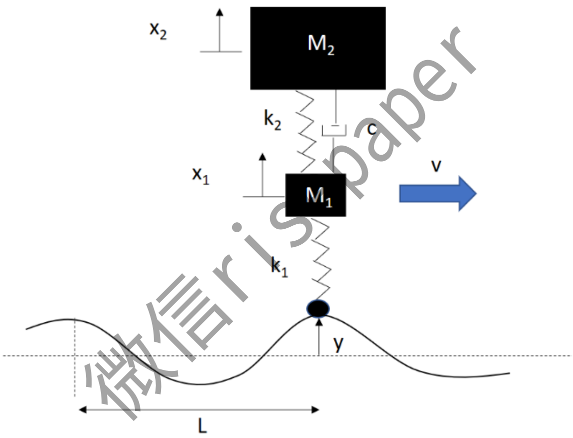
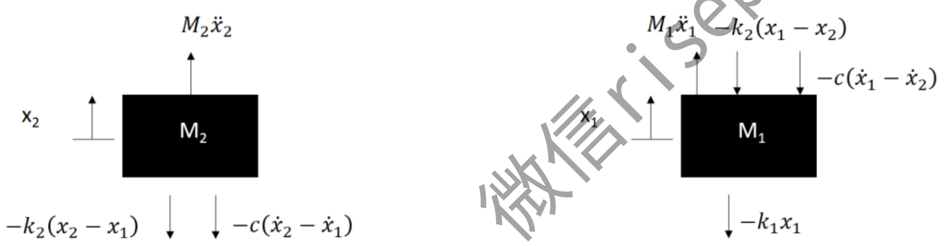


Figure Q1a **Figure Q1b**

Answer

a)



b)

$$M_2\ddot{x}_2 - c(\dot{x}_2-\dot{x}_1) - k_2(x_2-x_1) = 0 \qquad M_1\ddot{x}_1 - c(\dot{x}_1-\dot{x}_2) - k_2(x_1-x_2) - k_1x_1 = 0$$

c)

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

d)

mass matrix must be diagonal with positive entries,
stiffness matrix must be symmetric about diagonal with positive values along the leading diagonal

e)

$$\begin{bmatrix} k_1+k_2-M_1\omega^2 & -k_2 \\ -k_2 & k_2-M_2\omega^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$M_1M_2\omega^4 - M_1k_2\omega^2 - M_2k_1\omega^2 - M_2k_2\omega^2 + k_1k_2 = 0$$

$$M_1M_2\omega^4 - (M_1k_2 + M_2k_1 + M_2k_2)\omega^2 + k_1k_2 = 0$$

$$\omega^2 = \frac{(M_1k_2 + M_2k_1 + M_2k_2) \pm \sqrt{(M_1k_2 + M_2k_1 + M_2k_2)^2 - 4M_1M_2k_1k_2}}{2M_1M_2}$$

$$A = M_1M_2 = 2500$$
$$B = (M_1k_2 + M_2k_1 + M_2k_2) = 51000000$$
$$C = (k_1k_2) = 10000000000$$

$$\omega^2 = \frac{(B) \pm \sqrt{(B)^2 - 4AC}}{2A}$$

$$\omega_1 = 14.07\frac{rad}{s}$$
$$\omega_2 = 142.14\frac{rad}{s}$$

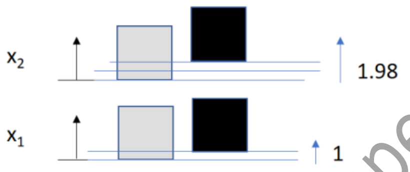
f)

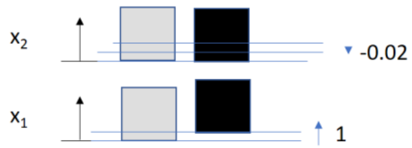
$$\begin{bmatrix} k_1 + k_2 - M_1\omega^2 & -k_2 \\ -k_2 & k_2 - M_2\omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\frac{x_2}{x_1} = \frac{k_1 + k_2 - M\omega^2}{k_2}$$

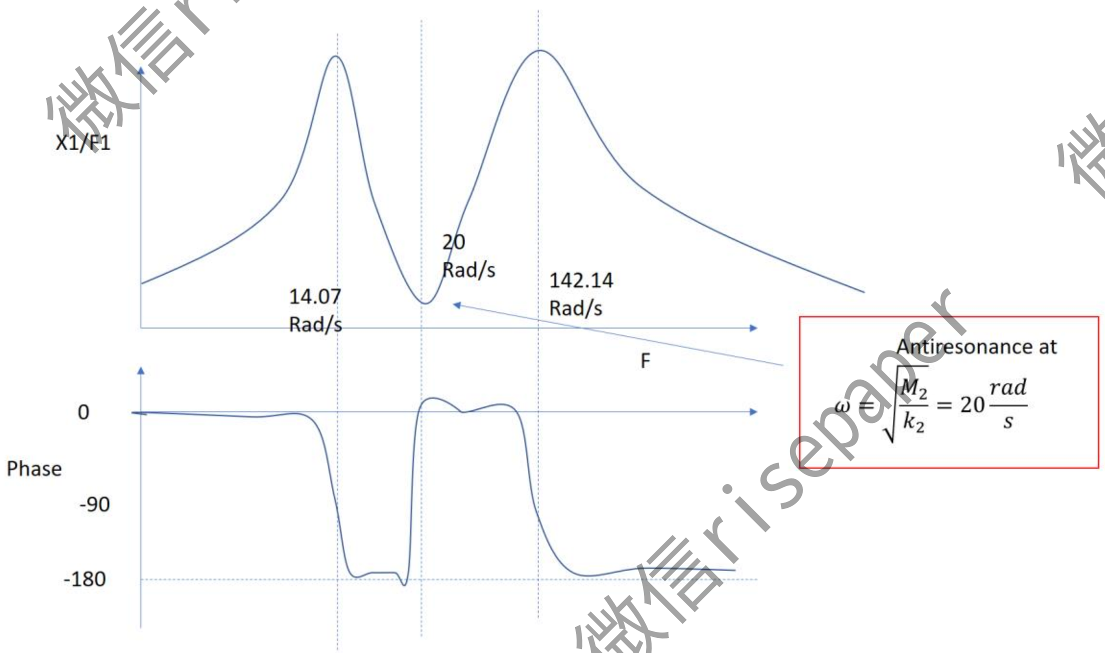
$$\omega_1 = 14.07\frac{rad}{s} \qquad \omega_2 = 142.14\frac{rad}{s}$$

$$\frac{x_2}{x_1} = \frac{k_1 + k_2 - M_1\omega_1^2}{k_2} = 1.98 \qquad \frac{x_2}{x_1} = \frac{k_1 + k_2 - M_1\omega_2^2}{k_2} = -0.02$$



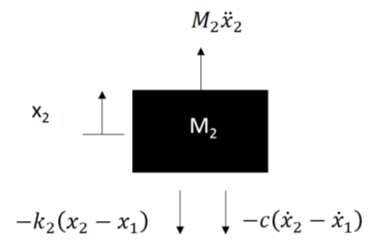


g)



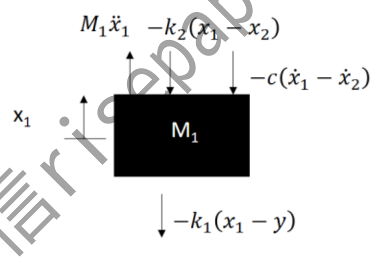
Turn over

h)



$$M_2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

Remains the same



$$M_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) + k_1x_1 = k_1y$$

i)

Natural frequency with most x2 motion is at omega1= 14.07 rad/s = 2.24 Hz, the other one is the wheel hop mode.

One cycle of oscillation is L =1m.

Therefore, velocity to excite at 2.24Hz is 2.24*1=2.24 m/s

2. Figure Q2a shows the fan of a ventilation system. The fan has mass M = 20kg and is centrally mounted on a rotor that is supported by two short bearings at its ends. The rotor is a solid steel rod of radius R = 5mm. It has a length L = 1m. There is concern that at some speeds excessive vibration of the fan could compromise the performance of the system.

   (a) Calculate the first critical speed of the fan disc-rotor assembly, assuming a massless rotor. [3%]

   (b) How would you expect your estimate in (a) to change if the mass of the rotor were taken into account? [1%]

   (c) A hammer test is to be performed on the real structure to measure the FRF of the system and to compare the measurement with your calculation. Sketch the system and indicate where you would best place an accelerometer to measure the response; also indicate where you would try to excite the structure with the hammer. [2%]

   (d) Figure Q2b shows the measured FRF. Assuming the system behaves like a single degree of freedom, estimate the actual damping ratio ζ, the actual mass M and stiffness k of the system from the graph. Explain your working. [6%]

   (e) At the critical speed the vertical displacement of the fan from the axis of rotation is observed to be v = 10mm. The bearings are supported by a small solid vertical member of cross-sectional area $A = 10^{-6}\text{m}^2$ as shown in Figure Q2a. Estimate the alternating axial stresses in the bearing support due to the rotor out of balance. [2%]

   (f) Using the S/N curve for the bearing support that is shown in Figure Q2c, estimate the fatigue life of the bearing supports at the critical speed assuming that the vibration results in purely axial stresses in the bearing supports. [2%]

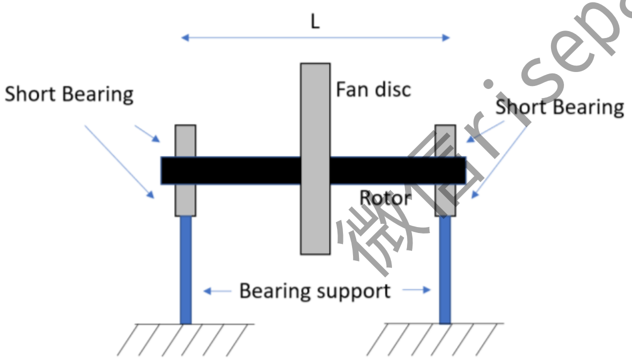


Figure Q2a

Turn over

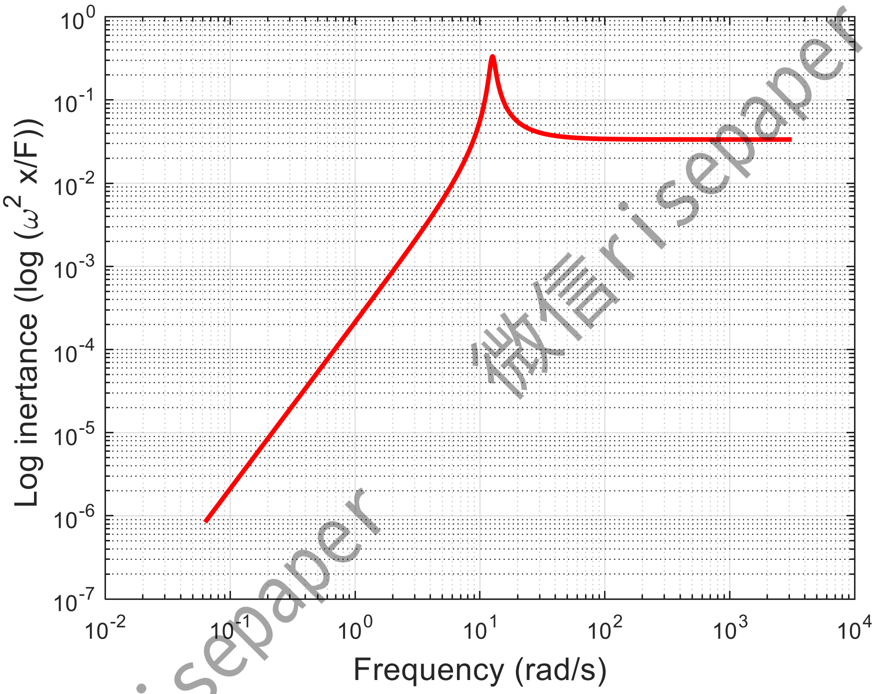


Figure Q2b

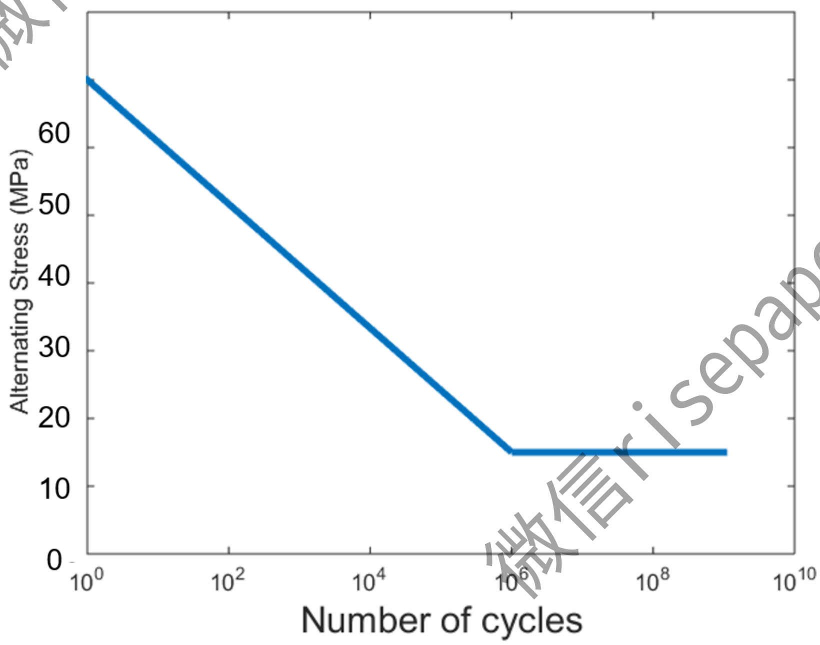


Figure Q2c

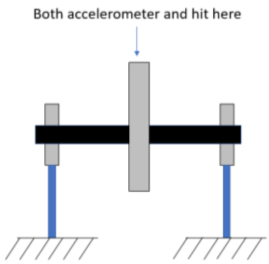
Answer Q2

a) Estimate critical speed by treating rotor as simply supported beam:

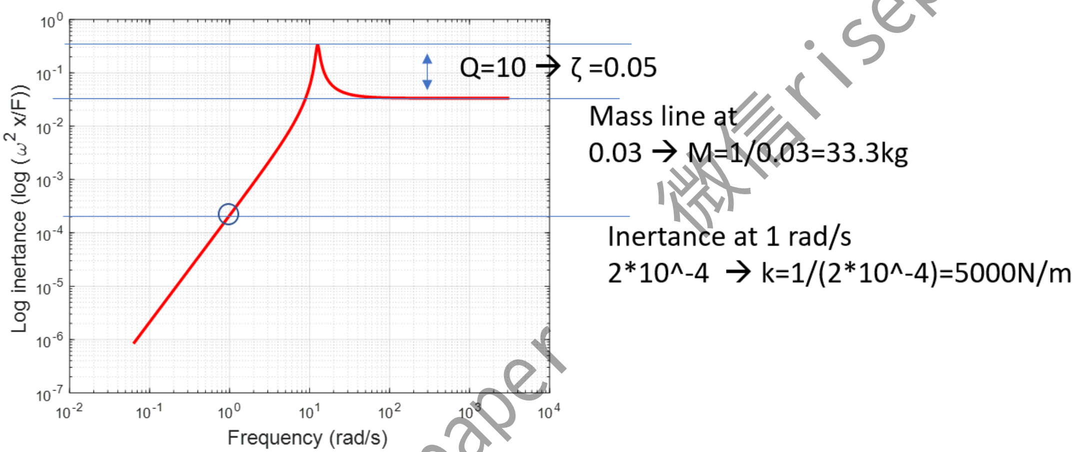
$$E = 200\text{GPa}$$
$$M = 20\text{kg}$$
$$I = \frac{\pi}{4}R^4 = 4.9 * 10^{-10} m^4$$
$$k = \frac{48EI}{L^3} = 4712.4\ N/m$$
$$\omega = \sqrt{\frac{k}{M}} = 15.4 \text{ rad/s} = 2.44 \text{ Hz}$$

b)
Stiffness unchanged but mass of rotor adds to mass of fan, hence increased overall mass and natural frequency and critical speed estimate is reduced.

c)



d)



e)
```
Force=M.*0.01*omega.^2 =47 N

Above is force that would be required to achieve deflection if
not at resonance, at resonance
Force= Force_static/Q= 47/10= 4.7 N
```

Turn over

```
 There are 2 supports therefore
Force at support = 4.7/2 = 2.35N
A=0.000001;
sigma=Force./A = 2.35 MPa
```

f)
Stress below the endurance limit -→ infinite life if no other degradation mechanism give rise to stress concentrations.

3. Figure Q3a shows the schematic of a tweeter type loudspeaker. It consists of a thin membrane that deflects horizontally and radiates sound. For the purposes of this question you can model this membrane as a beam. A coil of very thin wire is attached to the centre of the beam and a magnet is placed next to it. You may assume that the coil is mass-less except where otherwise stated. The centre of the beam experiences a force if a current is passed through the coil. The natural frequencies of the first three modes of vibration of the system are to be analysed. You may assume that the general form of the deflection *v* of the beam is:

$$v = C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x$$

(a) State all the boundary conditions that are required to model the flexural vibration of the beam in figure Q3a in form of equations. [2%]

(b) Using the boundary conditions of (a) derive the frequency equation for the flexural vibration of the beam. [4%]

(c) Sketch the mode shape of the first 3 modes of flexural vibration for the system of (a) - (b). [2%]

(d) For a beam of L=0.01m, $I=8.3{*}10^{-19}$ m^4, $A=10^{-7}$m^2, E= 40GPa, ρ=1000kg/m^3 determine the natural frequencies of the first three modes of flexural vibration of the above system. You may use the information in table Q3 to help you. [2%]

(e) The coil is modelled as mass-less, however in real life it has some mass. Discuss the effect of an additional mass in the centre of the beam on the vibration response. [2%]

(f) In addition to the effect discussed in e), there are other reasons for which the natural frequencies are likely to be different. Explain why and state whether you would expect the natural frequencies to be higher or lower than the calculated values. [2%]

(g) The speaker is designed to operate over a frequency range of 100-5000Hz. Because of space constraints in the housing a design engineer suggests that it would be beneficial to move the magnet and coil off-centre, as indicated in Figure Q3b. Comment on the effect of this design change on the vibration response of the beam. [2%]

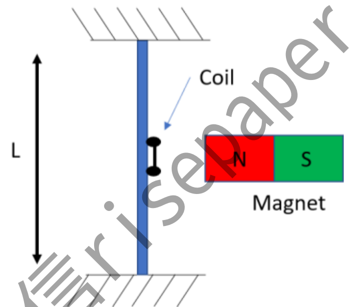


Figure Q3a

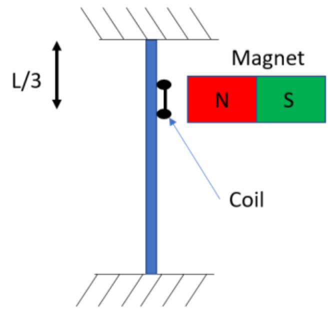


Figure Q3b

Turn over

Table of α values for the first 3 modes of systems with the below frequency equations:

With nat. frequency: $\omega = \frac{\alpha}{L^2}\sqrt{\frac{EI}{A\rho}}$

Frequency Equation	Mode 1	Mode 2	Mode 3
$\cos\lambda L * \cosh\lambda L = -1$	3.52	22.4	61.7
$\sin\lambda L = 0$	9.87	39.5	88.9
$\tan\lambda L = \tanh\lambda L$	15.4	50.0	104.0
$\cos\lambda L * \cosh\lambda L = 1$	22.4	61.7	121.0
$\tan\lambda L = \tanh\lambda L$	0	15.4	50.0

Table Q3

Answer Q3

a)

$$at\ x = 0\ ,\ \ v = 0\ \text{ and } \frac{\partial v}{\partial x} = 0$$

$$at\ x = L, v = 0 \text{ and } \frac{\partial v}{\partial x} = 0$$

b)

$$v(x) = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) + C_3 \cos(\lambda x) + C_4 \sin(\lambda x)$$

$$X = (C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x)$$

$$\frac{dX}{\partial x} = \lambda(C_1 \sinh \lambda x + C_2 \cosh \lambda x - C_3 \sin \lambda x + C_4 \cos \lambda x)$$

$$\frac{d^2X}{\partial x^2} = \lambda^2(C_1 \cosh \lambda x + C_2 \sinh \lambda x - C_3 \cos \lambda x - C_4 \sin \lambda x)$$

$$\frac{d^3X}{\partial x^3} = \lambda^3(C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x - C_4 \cos \lambda x)$$

$$X = (C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x)$$

$$at\ x = 0, v = 0 \text{ and } \frac{\partial v}{\partial x} = 0$$

$$C_1 + C_3 = 0 \Rightarrow C_3 = -C_1$$

$$C_2 + C_4 = 0 \Rightarrow C_4 = -C_2$$

$$at\ x = L, v = 0 \text{ and } \frac{\partial v}{\partial x} = 0$$

$$0 = (C_1 \cosh \lambda L + C_2 \sinh \lambda L - C_1 \cos \lambda L - C_2 \sin \lambda x) \Rightarrow C_2 = -\frac{C_1(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))}$$

$$0 = \lambda(C_1 \sinh \lambda L + C_2 \cosh \lambda L + C_1 \sin \lambda x - C_2 \cos \lambda x)$$

$$0 = \left(C_1 \sinh \lambda L - \frac{C_1(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))} \cosh \lambda L + C_1 \sin \lambda L + \frac{C_1(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))} \cos \lambda x\right)$$

Turn over

$$0 = \left(C_1 \sinh\lambda L - \frac{C_1(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))}\cosh\lambda L + C_1 \sin\lambda L + \frac{C_1(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))}\cos\lambda L\right)$$

$$\frac{(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))}\cosh\lambda L = (\sinh\lambda L + \sin\lambda L) + \frac{(cosh(\lambda L) - cos(\lambda L))}{(sinh(\lambda L) - sin(\lambda L))}\cos\lambda L$$

$$\cosh\lambda L(cosh(\lambda L) - cos(\lambda L)) = (\sinh\lambda L + \sin\lambda L)(sinh(\lambda L) - sin(\lambda L)) + \cos\lambda L(cosh(\lambda L) - cos(\lambda L))$$

$$(cosh^2(\lambda L) - cos(\lambda L)\cosh\lambda L) = (sinh^2(\lambda L) - sin^2(\lambda L)) + (\cos(\lambda L)cosh(\lambda L) - cos^2(\lambda L))$$

use $(cos(\lambda L))^2 + (sin(\lambda L))^2 = 1$

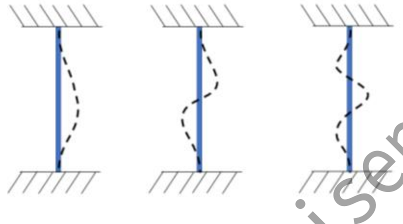
$$(cosh^2(\lambda L) - sinh^2(\lambda L)) = 2(\cos(\lambda L)cosh(\lambda L)) - 1$$

use $(cosh(\lambda L))^2 - (sinh(\lambda L))^2 = 1$

$$2 = 2(\cos(\lambda L)cosh(\lambda L))$$

$$\boxed{(\cos(\lambda L)cosh(\lambda L)) = 1}$$

c)



d)

$$\omega = \frac{\alpha}{L^2}\sqrt{\frac{EI}{A\rho}}$$

$$\omega_1 = \frac{22.4}{0.1^2}\sqrt{\frac{40*10^9*8*10^{-19}}{10^{-7}*1000}} = 4090 \text{ rad/s} = 650.9 \text{ Hz}$$

$$\omega_2 = \frac{61.7}{0.1^2}\sqrt{\frac{40*10^9*8*10^{-19}}{10^{-7}*1000}} = 11265 \text{ rad/s} = 1793 \text{ Hz}$$

$$\omega_3 = \frac{121}{0.1^2}\sqrt{\frac{40*10^9*8*10^{-19}}{10^{-7}*1000}} = 22091 \text{ rad/s} = 3516 \text{ Hz}$$

e)

Added mass at the centre reduces the natural frequency because of $\omega = \sqrt{\frac{k}{m}}$. For a thin membrane the added mass can be considerable and hence relatively large reductions from the predicted values in d) are to be expect.

f) Most likely the ends will not be completely built in and some rotation of the material will be possible, this means the natural frequencies will shift from the built-in ones towards those for pinned ends. Effectively the beam is less stiff, and the natural frequency reduces.

g) In the original design Mode 2 is not excited strongly because it has a node at the centred. The new off-centred design does now excite Mode 2. Mode two has a natural frequency at 1793Hz, i.e. very much in the range of interest (100-5000Hz). This means that the performance of the speaker can be considerably influenced by the design change.

4. Answer the following questions that are concerned with signal acquisition and processing:

(a) Explain the purpose of an anti-aliasing filter and illustrate its transfer function by means of a sketch. [3%]

(b) The Matlab code below contains an error. State the error and suggest an alternative statement that would fix the problem. [2%]

```
Fs=44100; %define sampling rate (Hz)
dt=1/Fs;  %define time step (seconds)
Y = recorded_data;  %data that was previously recorded
T = [ 0 : dt: (length(Y)-1)*dt];  %time vector
Spectrum = fft(Y);
Freq = [0:dt: (length(Y)-1).*dt];
Plot(Freq(1:end/2), abs(Spectrum(1:end/2))); % plot result
```

(c) The Matlab code below performs a particular operation on a recorded dataset after Fourier transformation. What is the code doing? [2%]

```
Fs=44100; %define sampling rate (Hz)
dt=1/Fs;  %define time step (seconds)
Y = recorded_data;  %data that was previously recorded
T = [ 0 : dt: (length(Y)-1)*dt];  %time vector
X = [1:length(Y)];
W = 1/2 .* (1 - cos(2.*pi.*X./(length(Y)-1)));
Y = Y.*W;
Spectrum = fft(Y);
Y(1)= 0;
Freq = [0:1./T(end): (length(Y)-1)./T(end)];
Plot(Freq(1:end/2), abs(Spectrum(1:end/2))); % plot result
```

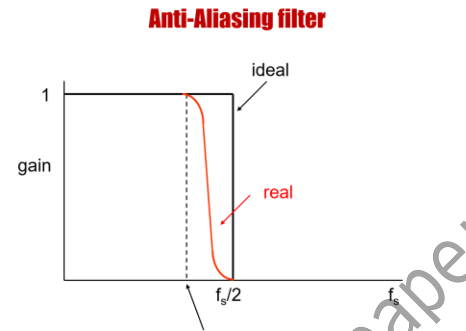
(d) You use a sampling rate of fs = 44 kHz to record a sound that contains two signals with frequency content of 300 and 330Hz. How many points do you need to record to tell the two signals apart in the spectrum? [2%]

(f) Define the following terms that are related to vibration measurements and signal processing:

(i) define the term “Inertance”, [1%]
(ii) define the term “Q-factor”, [1%]
(iii) describe what a “Pre-trigger” is, [1%]
(iv) explain what “Spectral leakage” is, and [2%]
(v) explain what the term “Quantisation level” means. [2%]

Answer Q4

a) The purpose of an anti-aliasing filter is to prevent any frequency higher than the Nyquist frequency from being recorded by an A/D converter. This avoids aliasing in any subsequent spectral analysis.



Often only display up to ~ 0.8 $f_s/2$ (e.g. line 400 out of 512)

b) The error is in the below line:

```
Freq = [0:dt: (length(Y)-1).*dt];
```

The frequency vector has interval df=1/T instead of dt. Therefore a correct statement would be:

```
Freq = [0:1./T(end): (length(Y)-1).*1./T(end)];
```

c) The script sets the first value in the fft vector equal to zero. This effectively removes the DC offset or mean of the signal.

d) df=30Hz, df=1./T therefore T >=0.0333 seconds fs =44000 points per second hence at least 1467 points need to be recorded. It is advisable to at least double that to about 3000 points.

e)

(i) Inertance is a particular format of the frequency response function (FRF) that displays acceleration response per unit excitation force.

(ii) the Q-factor is the ratio of “the displacement response at resonance” to “the displacement response under static conditions” for the same excitation force amplitude.

(iii) a Pre-trigger refers to a certain number of measurement points that are held in the memory of an A/D converter and which are recorded prior to a trigger event that starts a measurement. This is particularly useful in Hammer testing where a very short and sharp signal initiates the measurement and one does not want to lose the data that is measured during the very steep rise in force at the beginning of the measurement.

(iv) Spectral Leakage occurs when an FFT is calculated and the measured signal is not periodic within the measurement period. There is no spectral line available to represent the signal and hence the estimated spectrum contains energy in adjacent spectral lines. The energy is said to have leaked out into these adjacent spectral lines.

(v) A Quantisation level is the smallest voltage difference that an A/D converter can discern.

5. A robotic manipulator shown in Figure Q5a is powered by electric DC servomotors and Figure Q5b shows the control system for the shoulder joint. Forces due to gravity and dynamic coupling with other joints result in the external torque T experienced by the motor shaft. The total effective moment of inertia depends on the robot position within the working envelope in the range $0.002\,kgm^2 \le J \le 0.005\,kgm^2$. Initially G_c is set to be a proportional controller.

   (a) Derive the expression for the open loop transfer function *x/e* for $T = 0$. [2%]

   (b) Write the closed loop transfer function *x/u* and show that the system will have the smallest damping ratio for the largest value of inertia *J*. [3%]

   (c) Find the proportional controller gain that will result in the closed loop system damping ratio $\xi \ge 0.7$ throughout the robot working envelope. [3%]

   (d) For the gain in (c) and $J = 0.005\ kgm^2$ find the settling time to within 2% of the steady state position. [2%]

   (e) Figure 5Qc shows the root locus diagram for this system when G_c is a proportional-integral-derivative (PID) controller. Determine the closed loop poles that will result in damping ratio $\xi = 0.7$ and find the new settling time. [4%]

   (f) Comment on the relative performance of the proportional and the PID controller in terms of transient performance and steady state positioning accuracy. [2%]

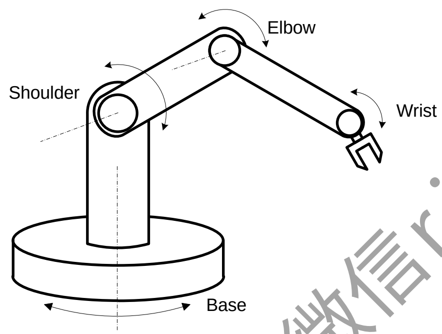


Figure Q5a

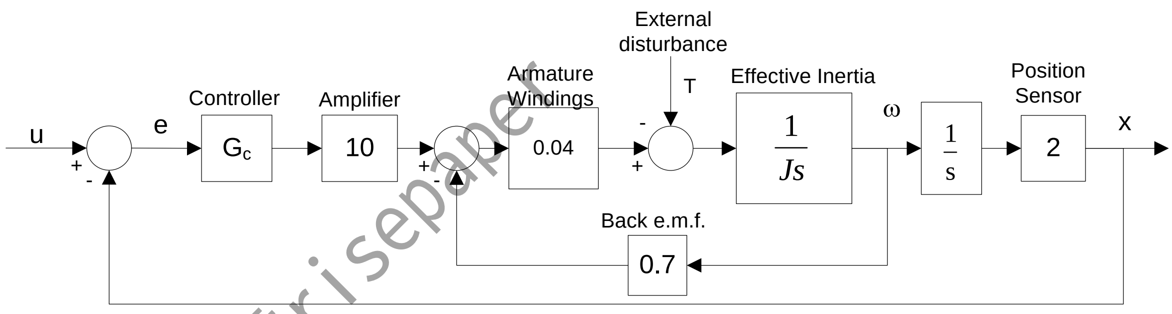


Figure Q5b

Turn over

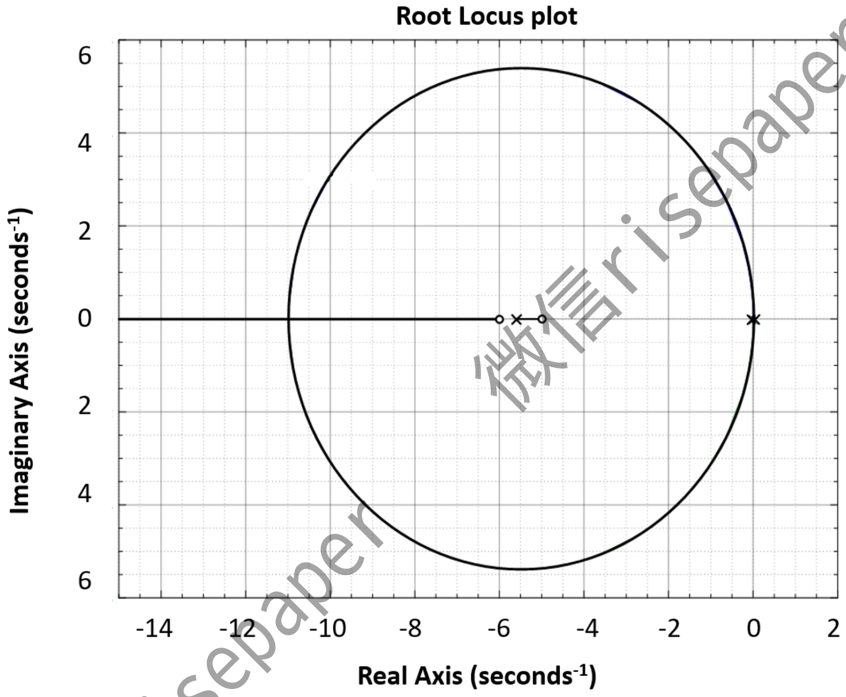

Root Locus plot
Imaginary Axis (seconds⁻¹)
6
4
2
0
2
4
6
-14
-12
-10
-8
-6
-4
-2
0
2
Real Axis (seconds⁻¹)

Figure Q5c

Answer Q5

a) $G_c = K$

$$G_{motor}(s) = \frac{\frac{0.04}{Js}}{1+\frac{0.04\,.\,0.7}{Js}} = \frac{0.04}{Js+0.028}$$

$$\frac{x}{e} = G(s) = (10)K(2)\left(\frac{0.04}{Js+0.028}\right).\left(\frac{1}{s}\right) = \boldsymbol{\frac{0.8K}{Js^2+0.028s}}$$

b)

$$C(s) = \frac{\frac{0.8K}{Js^2+0.028s}}{1+\frac{0.8K}{Js^2+0.028s}} = \boldsymbol{\frac{0.8K}{Js^2+0.028s+0.8K}} = \frac{\frac{0.8K}{J}}{s^2+\frac{0.028}{J}s+\frac{0.8K}{J}} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\omega_n = \sqrt{\frac{0.8K}{J}}$$

$$2\xi\omega_n = \frac{0.028}{J}$$

$$2\xi\sqrt{\frac{0.8K}{J}} = \frac{0.028}{J}$$

$$\boldsymbol{\xi = 0.0157\sqrt{\frac{1}{KJ}}}$$

Therefore ξ will be smallest for the largest J

c) $J = 0.005\ Nm^2$, $\xi = 0.7$

$$\xi^2 = 0.000247\frac{1}{KJ}$$

$$K = 0.000247\frac{1}{\xi^2 J} = 0.000247\frac{1}{0.49\ \times\ 0.005} = \boldsymbol{0.101}$$

d) Characteristic equation:

$$Js^2 + 0.028s + 0.8K = 0$$

Closed loop poles:

$$s = \frac{-0.028 \pm \sqrt{0.028^2 - 3.2JK}}{2J} = \frac{-0.028 \pm \sqrt{0.028^2 - 3.2(0.005)(0.101)}}{0.01} = \frac{-0.028 \pm j0.0288}{0.01}$$

$$\boldsymbol{s = -2.8 \pm j2.88}$$

Settling time to 2%: $\alpha = -2.8$

$e^{\alpha t} = 2\%$

$e^{-2.8t} = 0.02$

$-2.8t = -3.912$

$\boldsymbol{t = 1.40\ s}$

e) $\theta = \sin 0.7 = 44°$

Closed loop poles at $\boldsymbol{s = -5.3 \pm j5.4}$

Settling time to 2%: $\alpha = -5.3$

$e^{\alpha t} = 2\%$

$e^{-5.3t} = 0.02$

$-5.3t = -3.912$

$\boldsymbol{t = 0.74\ s}$

f) The PID controller results in about 2x faster transient response for the same damping ratio and overshoot.

P-control is expected to have finite positioning error due to gravity. PID control is expected to eliminate this error owing to the additional integration.

Turn over

6. The steering system of an experimental self-driving car was designed for automated cruising along the motorway. It uses a vision system to estimate the deviation x of the car from the centreline of the motorway lane. The signals *u*, *D*, *x* and *e* are measured in pixels of the look-ahead camera. The simplified closed loop control system is shown in Figure Q6. The control design requirements are that the system should have a phase margin PM=60°, and a corresponding gain margin GM≥10dB.

(a) What should be the value of u in normal cruising conditions? [2%]

(b) If C(s) is a simple proportional controller of gain K=1, sketch the Bode magnitude and phase diagrams in asymptotic form. [6%]

(c) Find the maximum gain K for which the closed loop system is stable. [2%]

(d) Find the gain K that would best meet the control design specification. [2%]

(e) The car approaches a constant radius bend in the road, which results in suddenly applying D=100 pixels. Find the steady state error when following the bend. [4%]

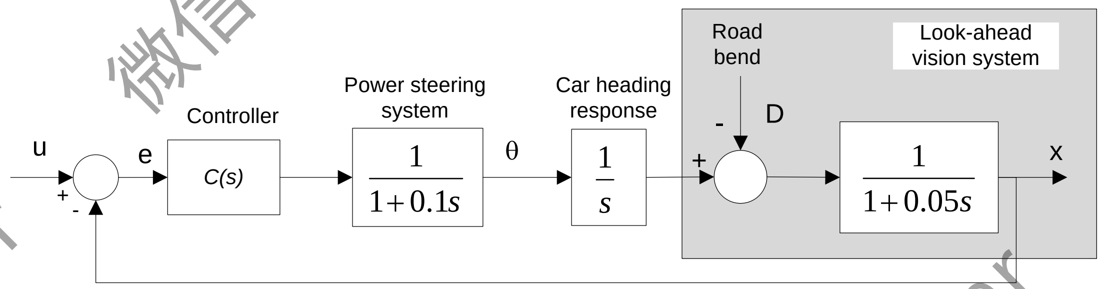


Figure Q6

Answer Q6

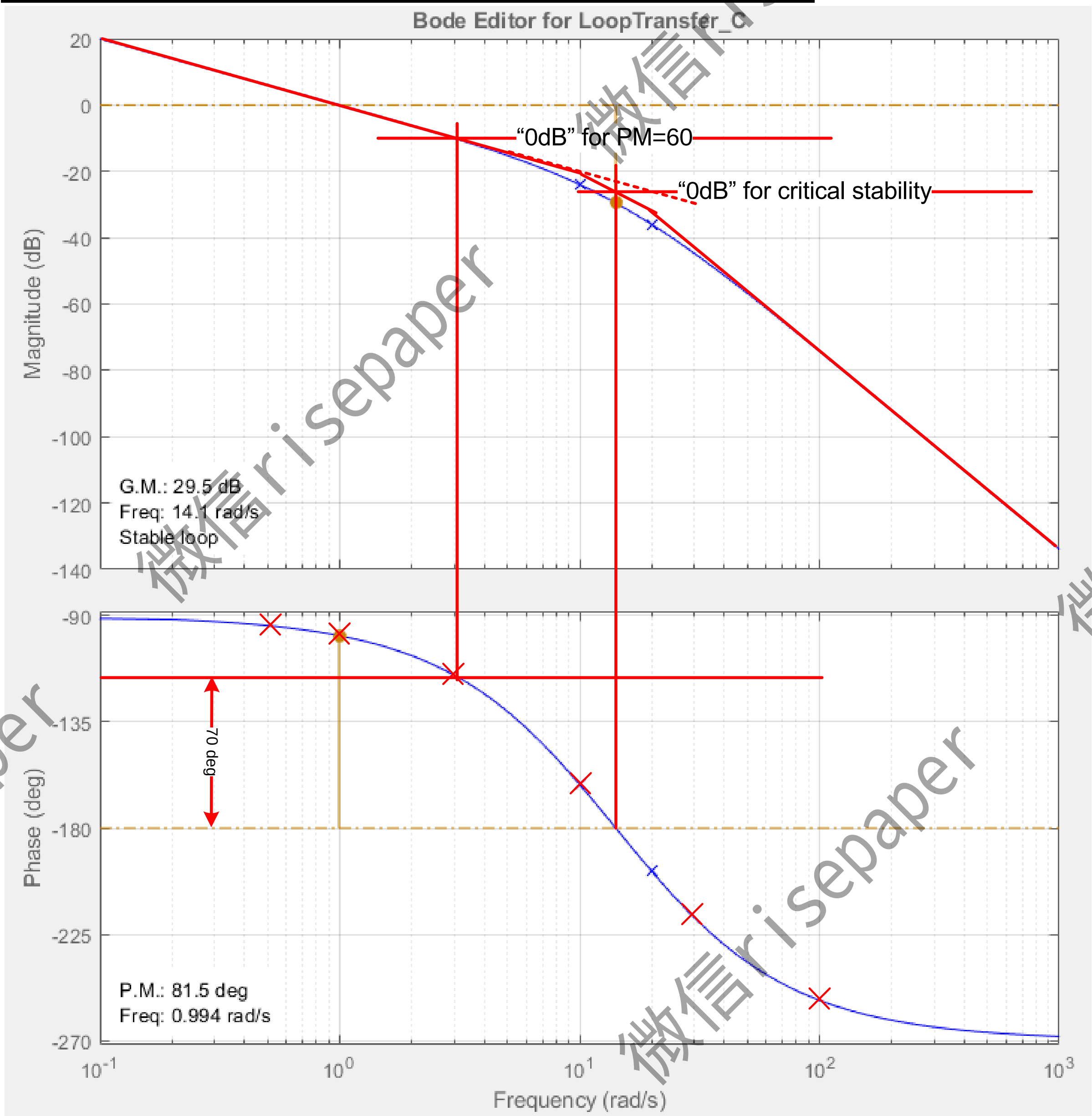
a) $u = 0$

b) Open loop transfer function: $G(s) = \frac{1}{s(1+0.1s)(1+0.05s)}$

Corner frequencies: $10\ rad/s,\ 20\ rad/s$

Data for phase plot: $\phi = -90 - \tan^{-1}(0.1\omega) - \tan^{-1}(0.05\omega)$

ω	0.5	1	3	10	30	100
ϕ	-94	-99	-115	-162	-218	-253



c) By construction $K_{max} = \omega = 20$

d) For $PM = 60$, construction gives $K = 3$

$GM \approx 12dB$, satisfies design requirements

e) $D(s) = \frac{100}{s}$, step function

$$x(s) = \frac{3}{s(1+0.1s)}e(s) - \frac{1}{(1+0.05s)}D(s)$$
$$e(s) = u(s) - x(s) = -x(s)$$
$$-e(s) = \frac{3}{s(1+0.1s)}e(s) - \frac{1}{(1+0.05s)}D(s)$$
$$-e(s)s(1+0.1s)(1+0.05s) = 3(1+0.05s)e(s) - s(1+0.1s)D(s)$$
$$e(s)\big(3(1+0.05s) + s(1+0.1s)(1+0.05s)\big) = s(1+0.1s)D(s)$$

Turn over

$$e(s) = \frac{s(1+0.1s)}{3(1+0.05s)+s(1+0.1s)(1+0.05s)}D(s)$$

$$E_{ss} = \lim_{s\to 0}\left(s\frac{s(1+0.1s)}{3(1+0.05s)+s(1+0.1s)(1+0.05s)}\left(\frac{100}{s}\right)\right)$$

$$= \lim_{s\to 0}\left(\frac{s(1+0.1s)}{3(1+0.05s)+s(1+0.1s)(1+0.05s)}(100)\right) = 0$$